Time Response Analysis

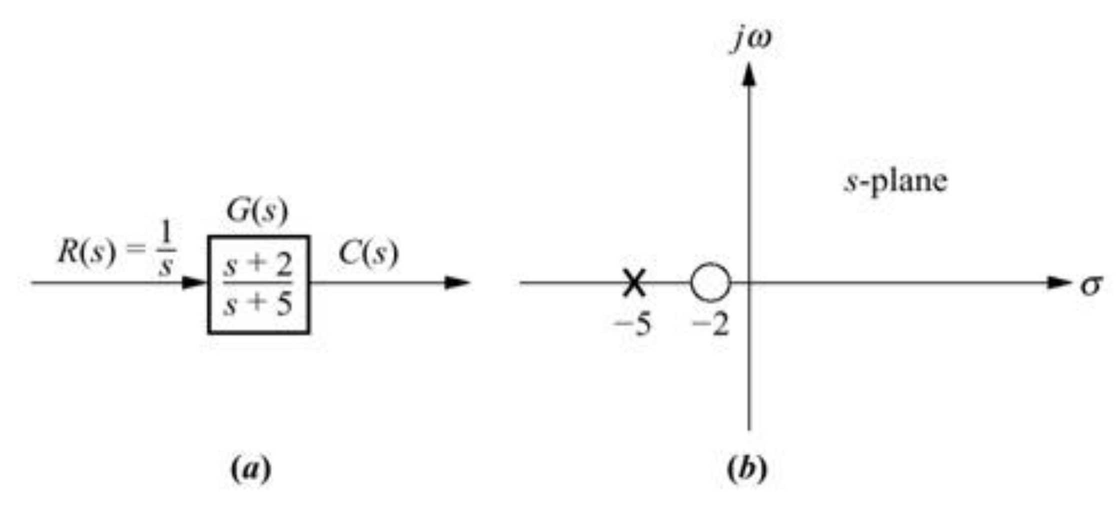
Contents

Introduction

- The concept of poles and zeros, fundamental to the analysis of and design of control system, simplifies the evaluation of system response.
- The poles of a transfer function are:
  i. Values of the Laplace Transform variables s, that cause the transfer function to become infinite.
  ii. Any roots of the denominator of the transfer function that are common to roots of the numerator.
- The zeros of a transfer function are:
  i. The values of the Laplace Transform variable s, that cause the transfer function to become zero.
  ii. Any roots of the numerator of the transfer function that are common to roots of the denominator.

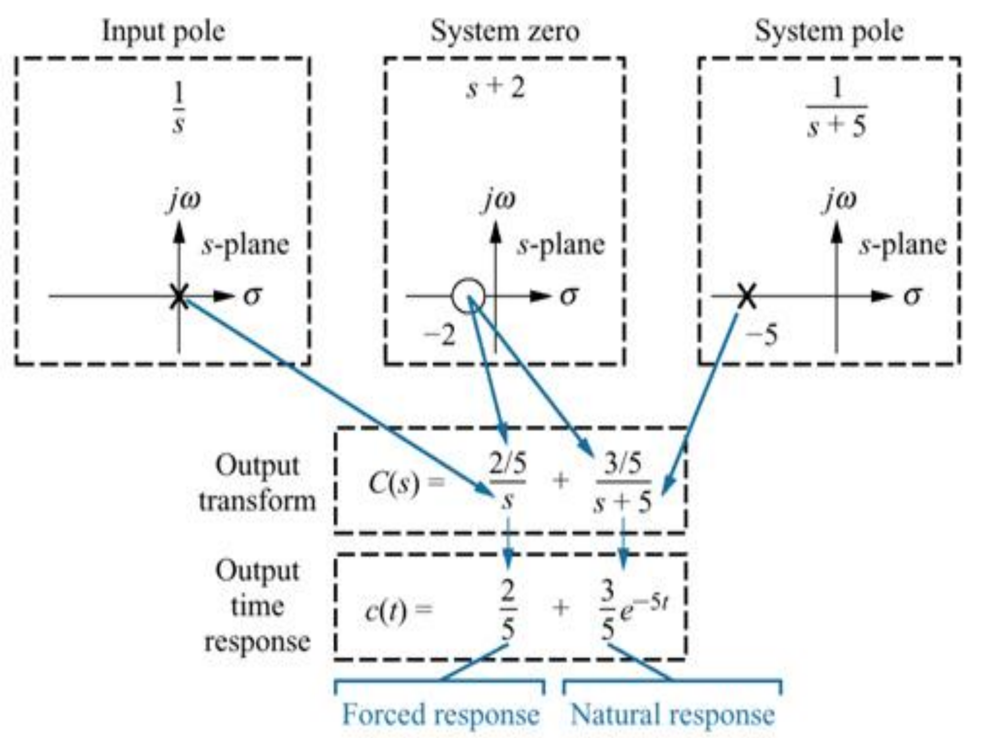
Influence of Poles on Time Response

- The output response of a system is a sum of
  i. Forced response
  ii. Natural response



a) System showing an input and an output
b) Pole-zero plot of the system

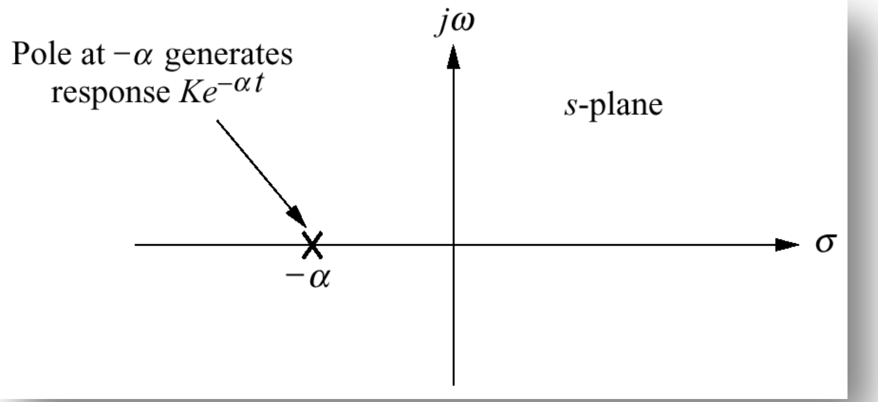
Influence of Poles on Time Response



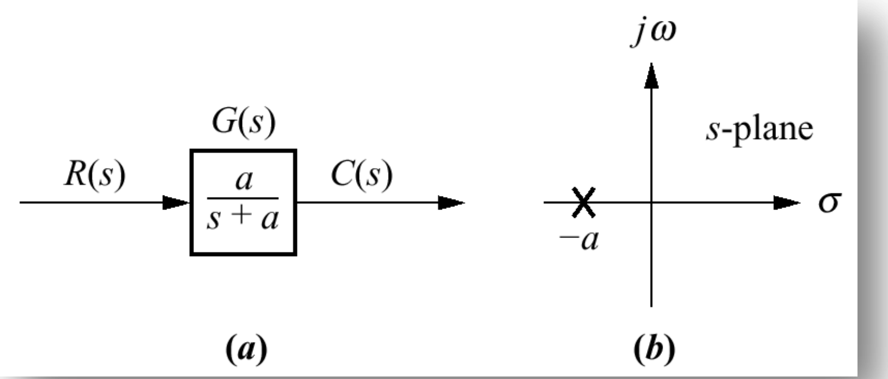
(c)

c) Evolution of a system response. Follow the blue arrows to see the evolution of system component generated by the pole or zero

Influence of Poles on Time Response



Effect of a real-axis pole upon transient response



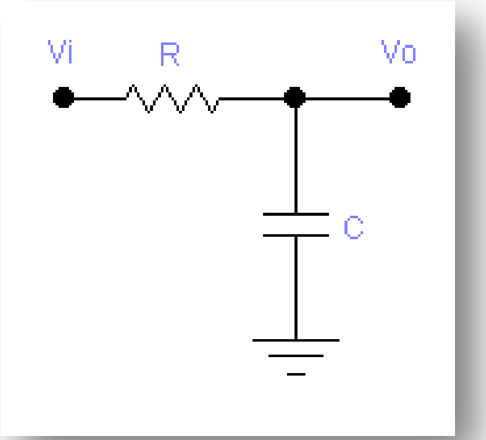
a) First-order system
b) Pole plot of the system

First-Order System

- General form:

$$G(s)=\frac{C(s)}{R(s)}=\frac{K}{\tau s+1}$$

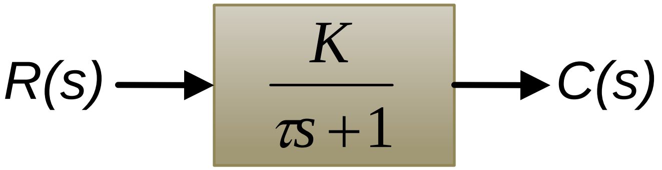
- Problem: Derive the transfer function for the following circuit



$$G(s)=\frac{1}{RCs+1}$$

First-Order System

- Transient Response: Gradual change of output from initial to the desired condition.
- Block diagram representation:



Where,
K : Gain
τ : Time constant

- By definition itself, the input to the system should be a step function which is given by the following:

$$R(s) = \frac{1}{s}$$

First-Order System

- General form:

$$G(s) = \frac{C(s)}{R(s)} = \frac{K}{\tau s + 1} \quad \Rightarrow \quad C(s) = G(s)R(s)$$

- Output response:

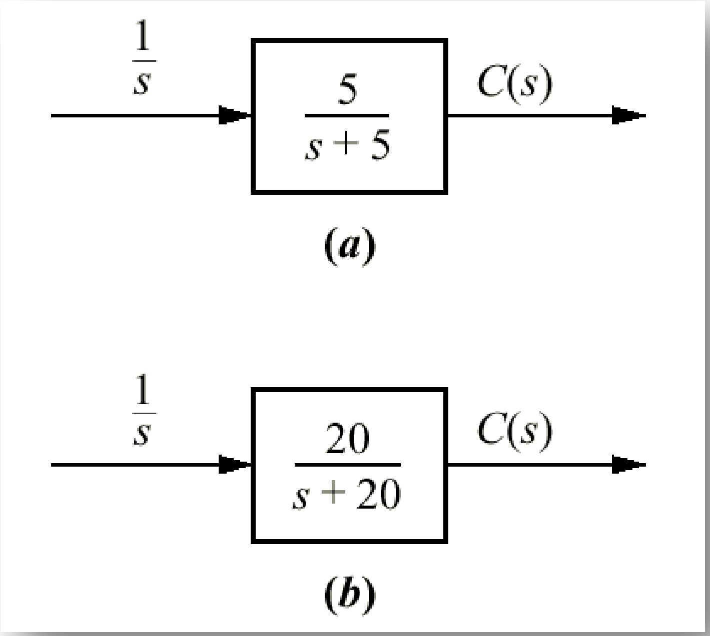
$$C(s) = \left(\frac{1}{s}\right)\left(\frac{K}{\tau s + 1}\right)$$

$$= \frac{A}{s} + \frac{B}{\tau s + 1}$$

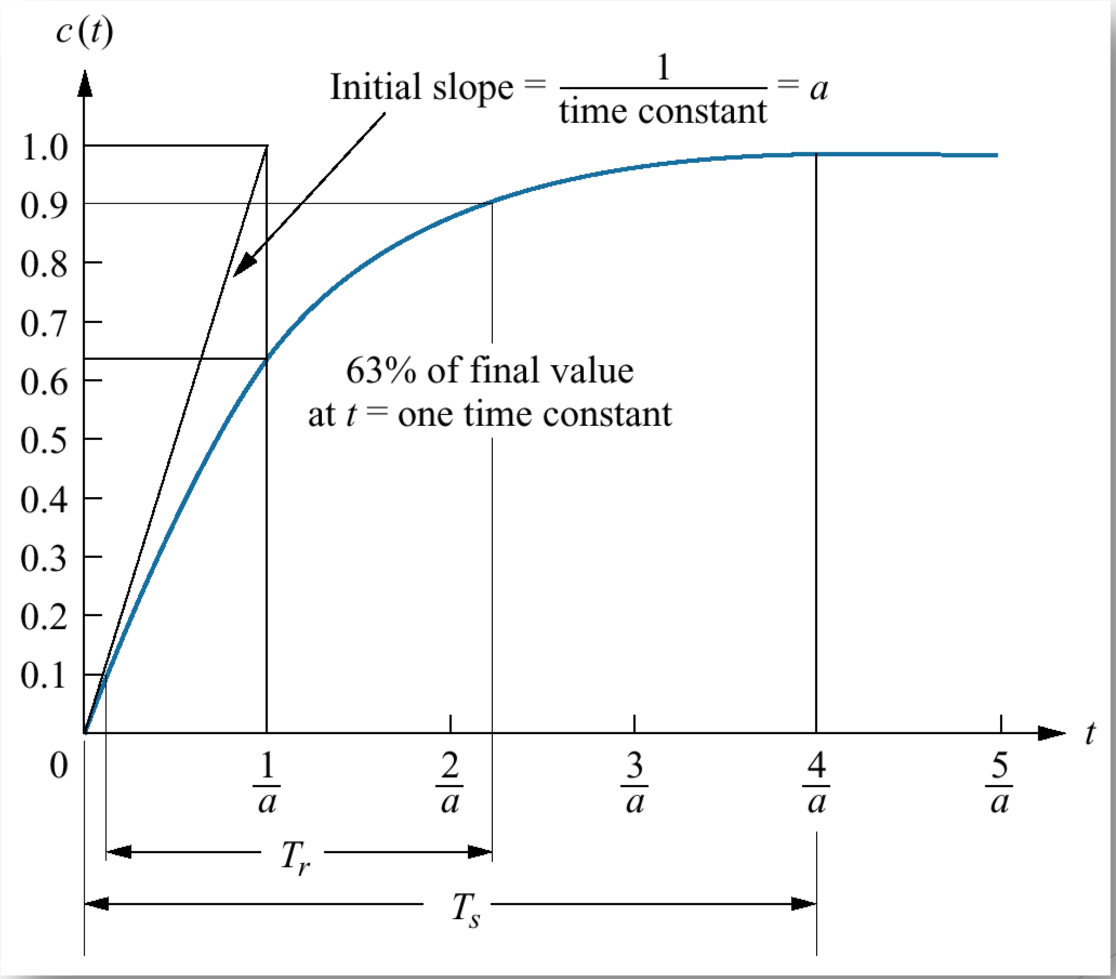
$$c(t) = A + \frac{B}{\tau} e^{-t/\tau}$$

First-Order System

- Problem: Find the forced and natural responses for the following systems



First Order System



First-order system response to a unit step

Transient Response Specifications

- Time constant, τ
  - The time for e^{-at} to decay 37% of its initial value.

$$\tau = \frac{1}{a}$$

- Rise time, t_r
  - The time for the waveform to go from 0.1 to 0.9 of its final value.

$$t_r = \frac{2.2}{a}$$

- Settling time, t_s
  - The time for the response to reach, and stay within 2% of its final value.

$$t_s = \frac{4}{a}$$

Transient Response Specifications

- Problem: For a system with the transfer function shown below, find the relevant response specifications

$$G(s) = \frac{50}{s+50}$$

i. Time constant, τ
ii. Settling time, t_s
iii. Rise time, t_r

Second-Order System

- General form:

$$G(s)=\frac{K\omega_n^2}{s^2+2\varsigma\omega_n s+\omega_n^2}$$

Where,

K : Gain

ς : Damping ratio

ω_n : Undamped natural frequency

- Roots of denominator:

$$s^2+2\varsigma\omega_n s+\omega_n^2=0$$

$$s_{1,2}=-\varsigma\omega_n\pm\omega_n\sqrt{\varsigma^2-1}$$

Second-Order System

- Natural frequency, ω_n
  - Frequency of oscillation of the system without damping.

- Damping ratio, ς
  - Quantity that compares the exponential decay frequency of the envelope to the natural frequency.

$$\boxed{\varsigma = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/s)}}}$$

Second-Order System

- Problem: Find the step response for the following transfer function

$$G(s) = \frac{225}{s^2 + 30s + 225}$$

- Answer:

$$c(t) = 1 - e^{-15t} - 15te^{-15t}$$

Second-Order System

- Problem: For each of the transfer function, find the values of ς and ω_n, as well as characterize the nature of the response.

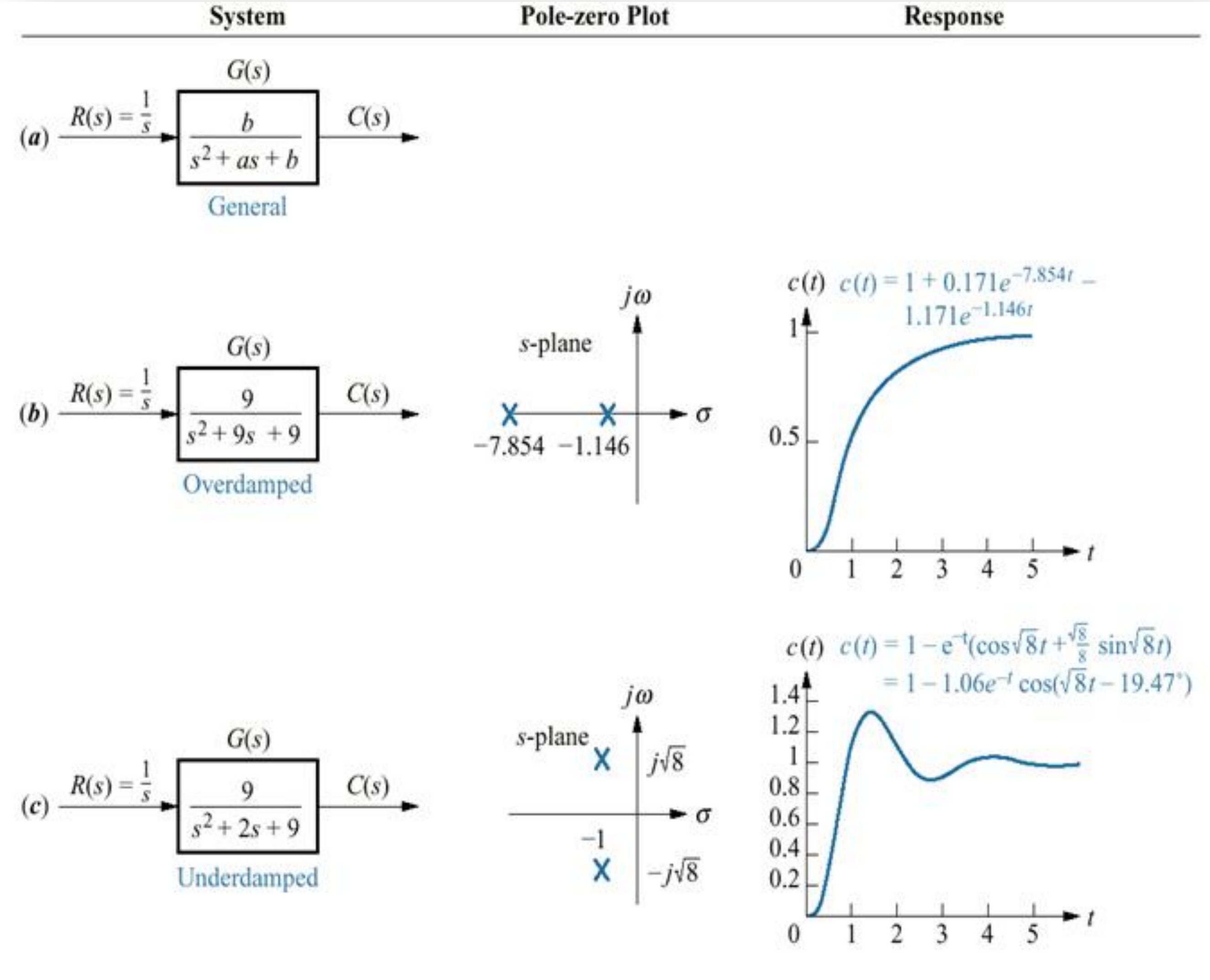
a) $$G(s) = \frac{400}{s^2 + 12s + 400}$$

b) $$G(s) = \frac{900}{s^2 + 90s + 900}$$

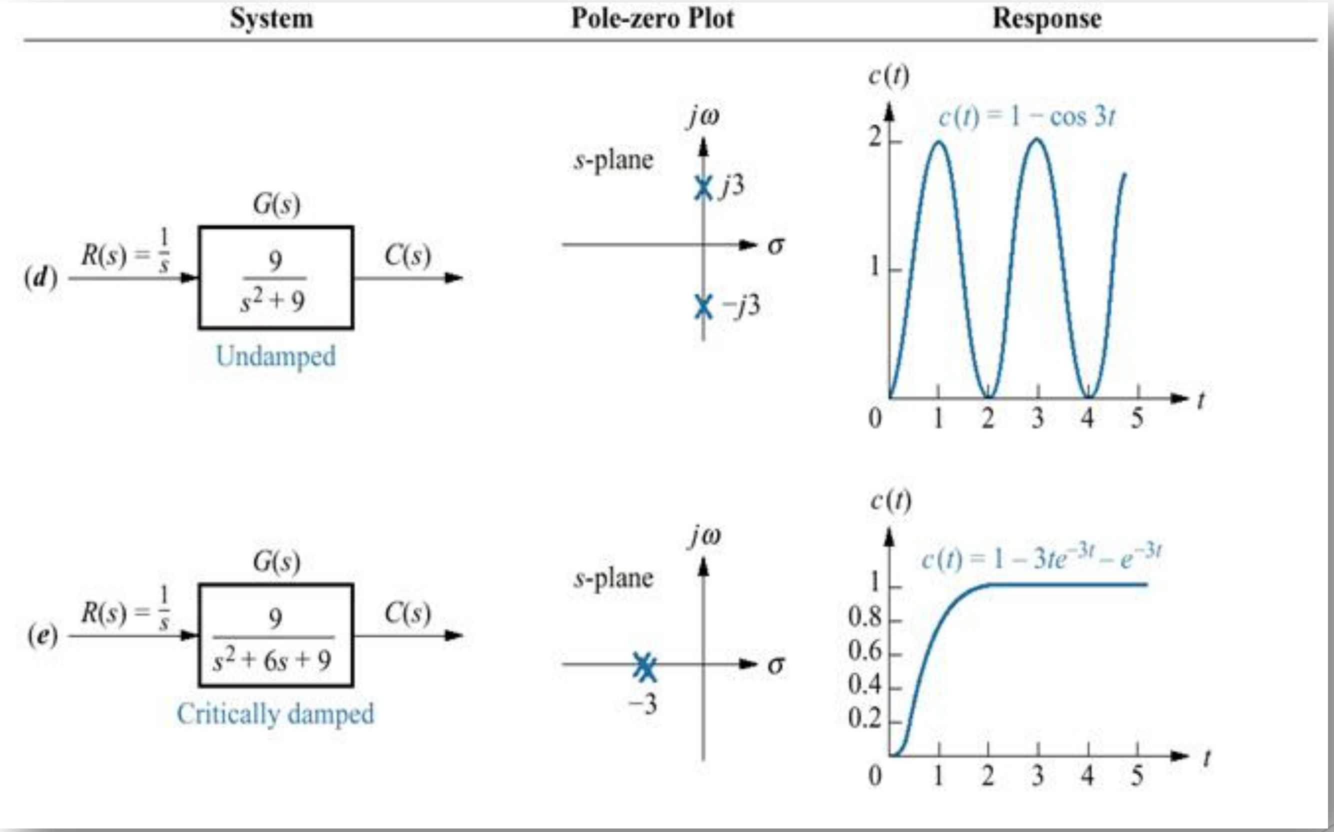
c) $$G(s) = \frac{225}{s^2 + 30s + 225}$$

d) $$G(s) = \frac{625}{s^2 + 625}$$

Second-Order System

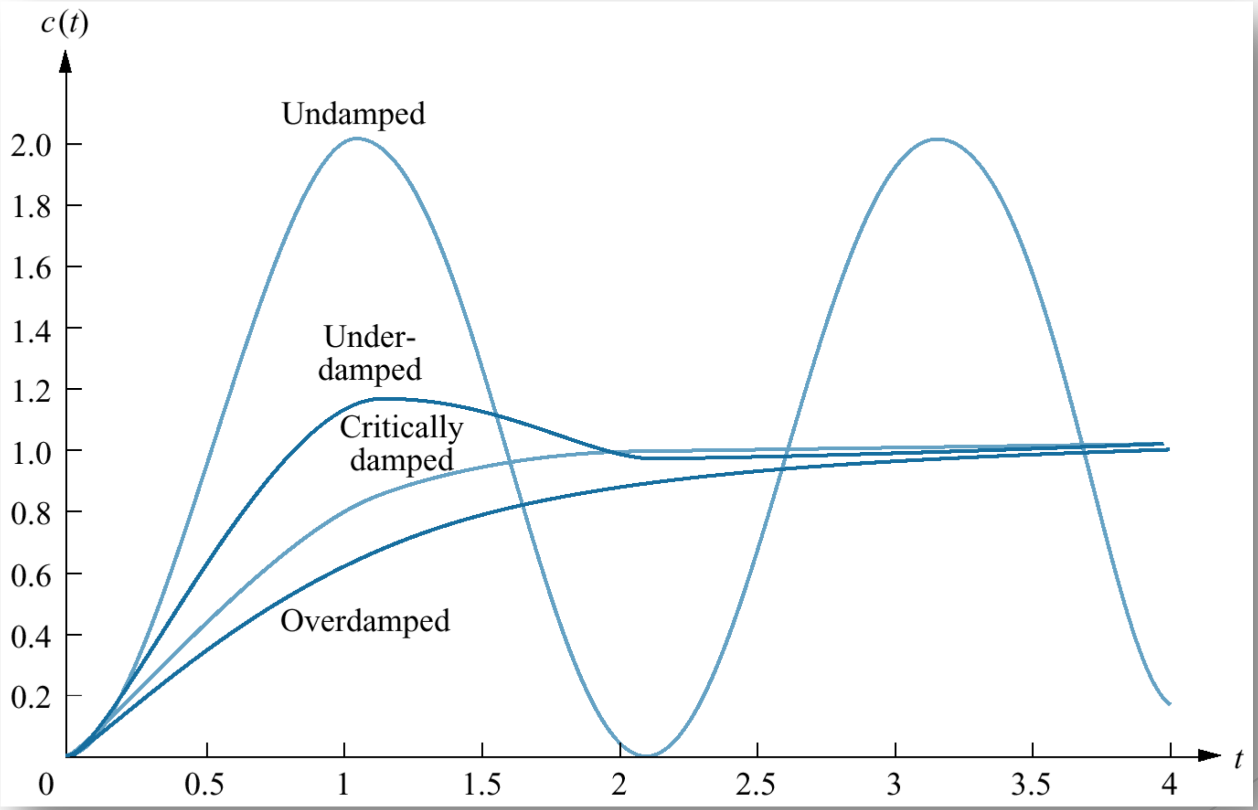


Second-Order System



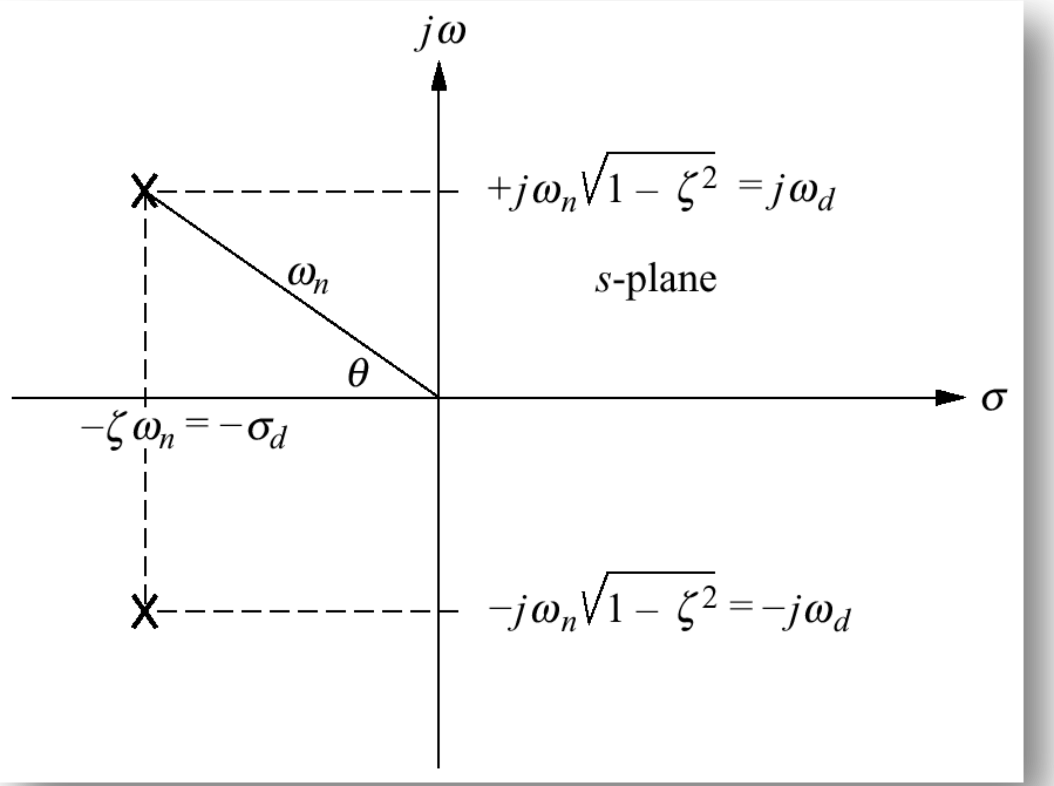
Second-Order System

- Step responses for second-order system damping cases



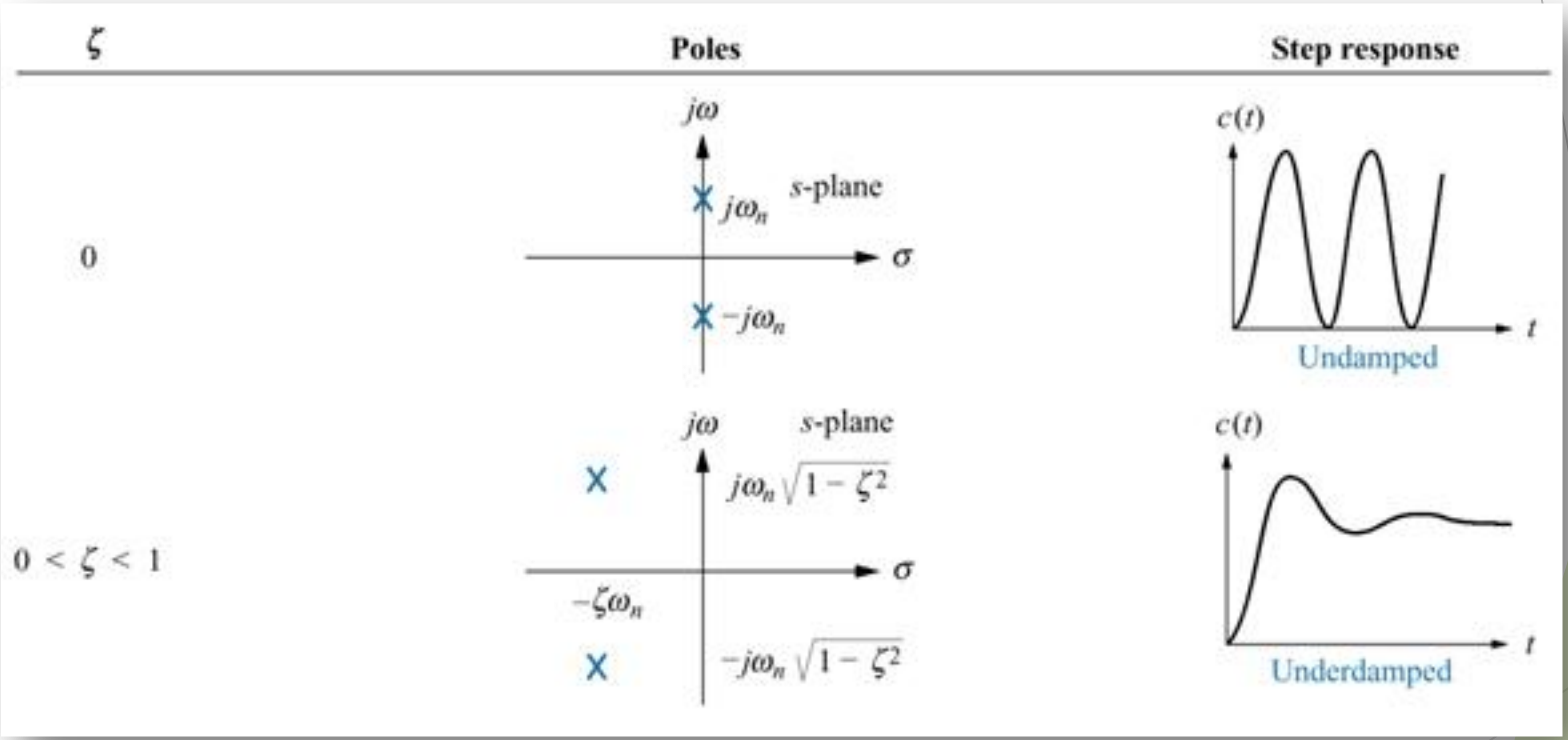
Second-Order System

- Pole plot for the underdamped second-order system



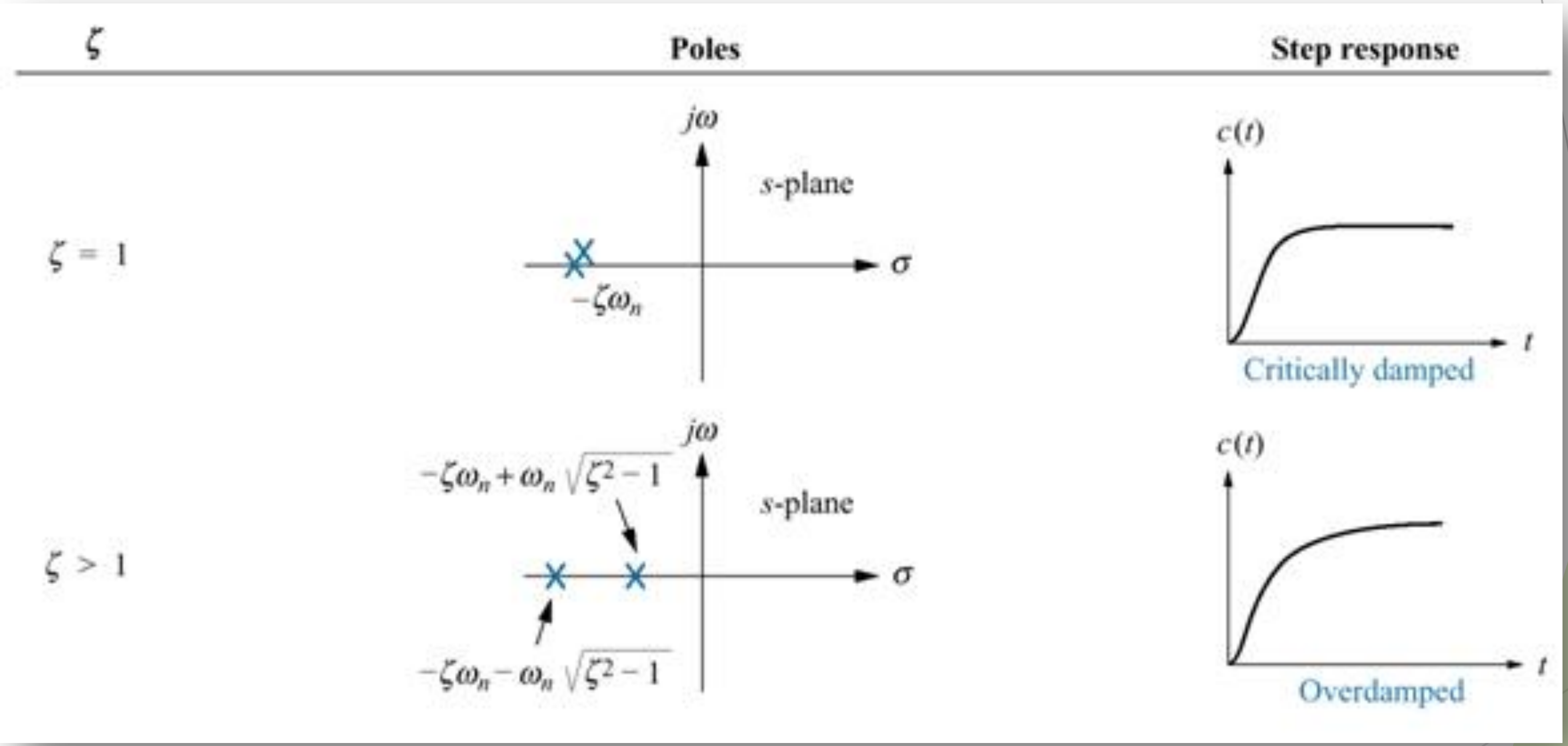
Second-Order System

- Second-order response as a function of damping ratio



Second-Order System

- Second-order response as a function of damping ratio



Second-Order System

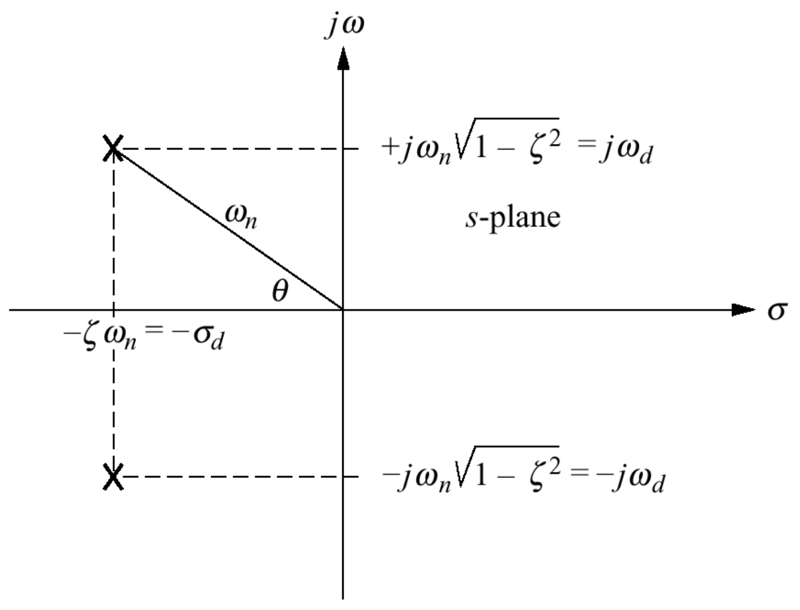
- When 0 < ς < 1, the transfer function is given by the following:

$$G(s)=\frac{K\omega_n^2}{(s+\varsigma\omega_n+j\omega_d)(s+\varsigma\omega_n-j\omega_d)}$$

Where,

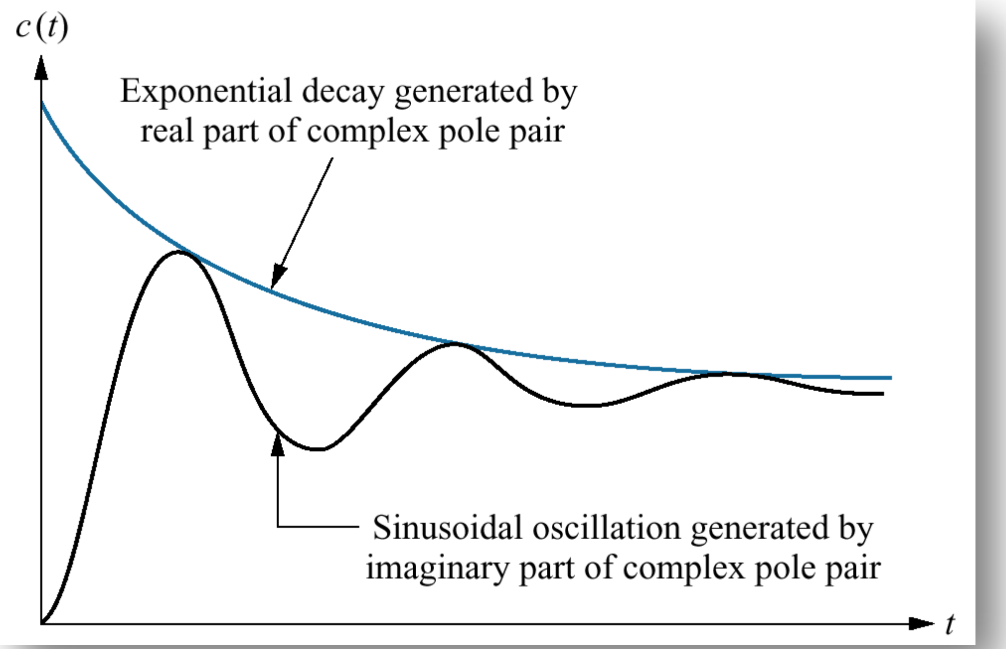
$$\omega_d=\omega_n\sqrt{1-\varsigma^2}$$

- Pole position:



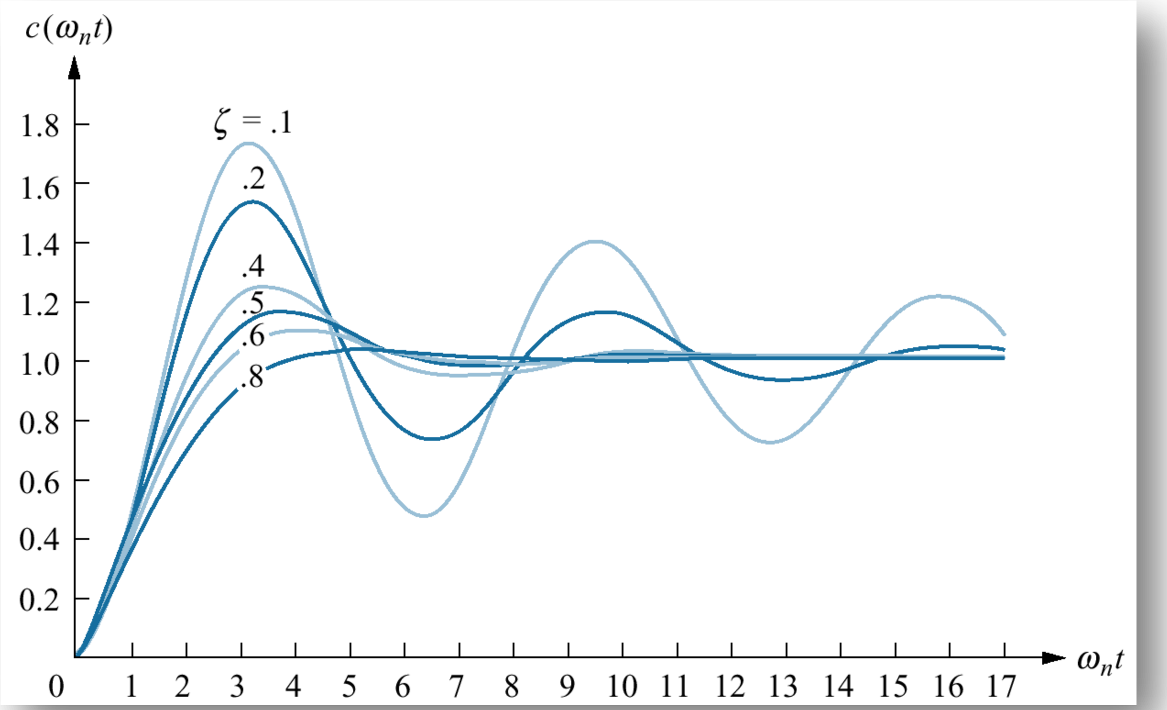
Second-Order System

- Second-order response components generated by complex poles



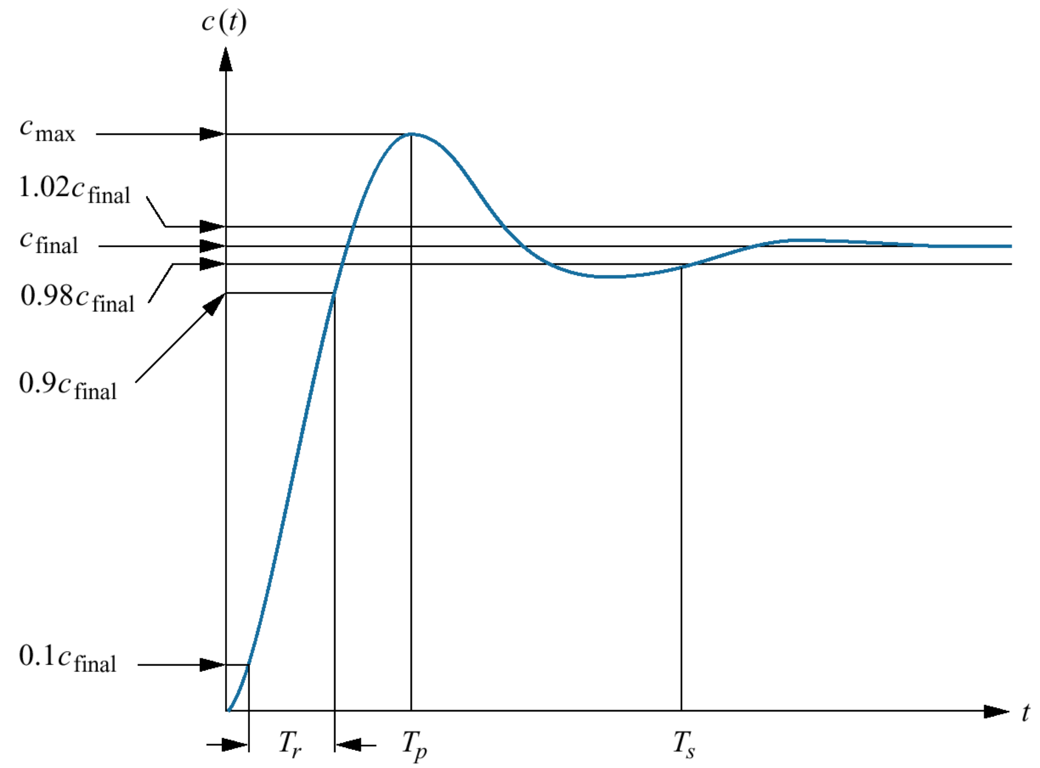
Second-Order System

- Second-order underdamped responses for damping ratio value



Transient Response Specifications

- Second-order underdamped response specifications



Transient Response Specifications

- Rise time, T_r
  - The time for the waveform to go from 0.1 to 0.9 of its final value.
- Peak time, T_p
  - The time required to reach the first or maximum peak.

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

- Settling time, T_s
  - The time required for the transient's damped oscillation to reach and stay within ±2% of the steady-state value.

$$T_s = \frac{4}{\zeta\omega_n}$$

Transient Response Specifications

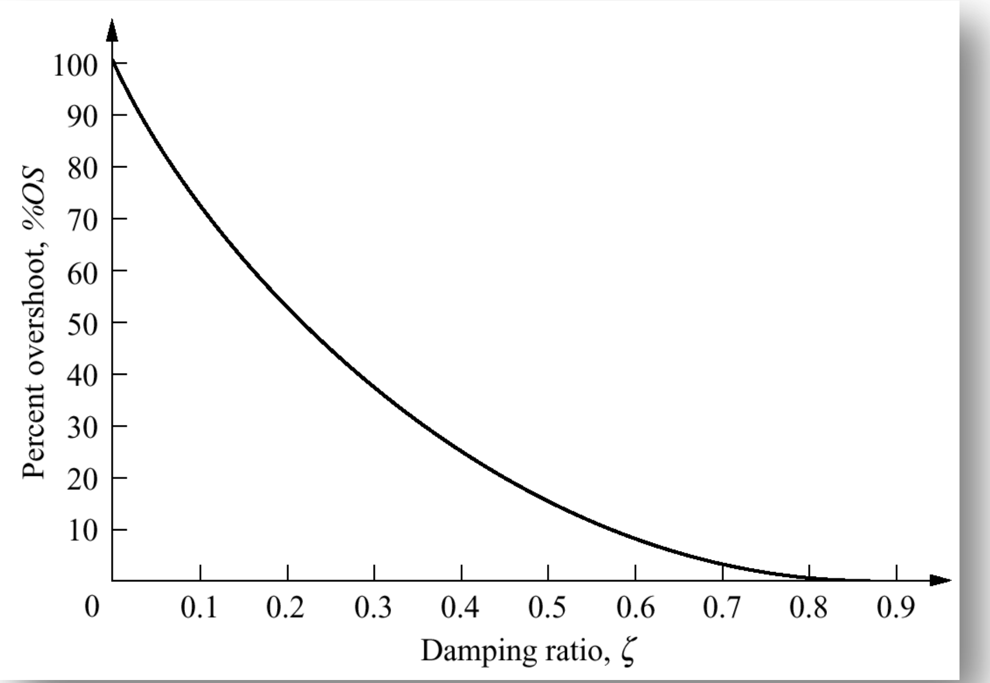
- Percent overshoot, %OS
  - The amount that the waveform overshoots the steady-state, or final value at peak time, expressed as a percentage of the steady-state value.

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100\%$$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

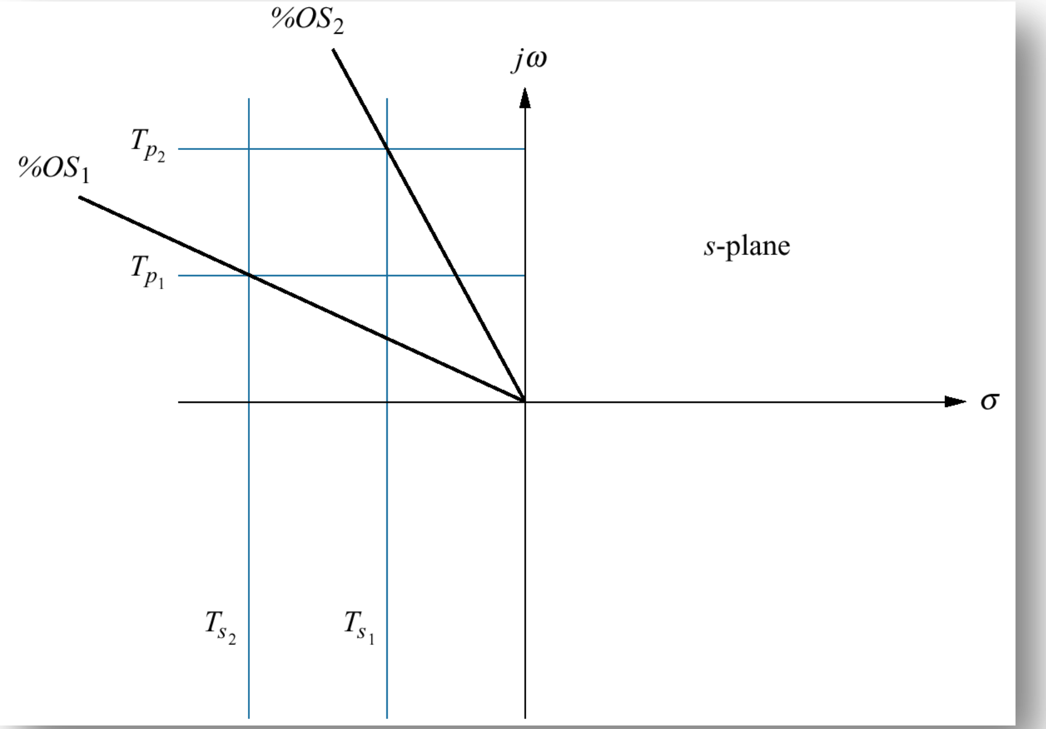
System Performance

- Percent overshoot versus damping ratio



System Performance

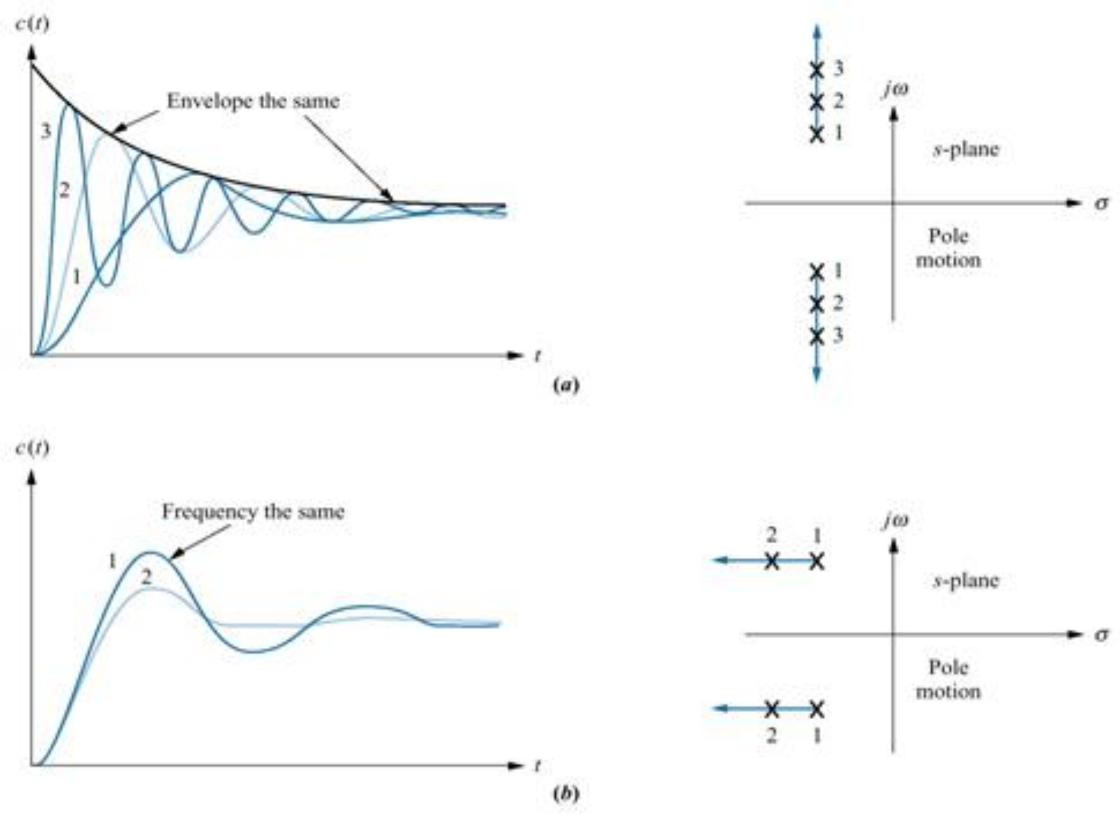
- Lines of constant peak time T_p, settling time T_s and percent overshoot %OS



$T_{s2} < T_{s1}$
$T_{p2} < T_{p1}$
$\%OS_1 < \%OS_2$

System Performance

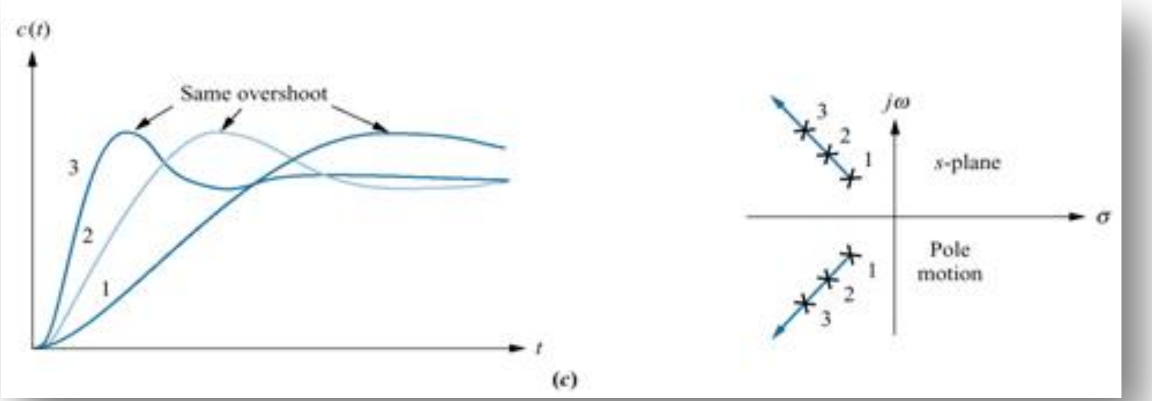
- Step responses of second-order underdamped systems as poles move



a) With constant real part
b) With constant imaginary part

System Performance

- Step responses of second-order underdamped systems as poles move



c) With constant damping ratio

Further Reading…

- Chapter 4
  i. Nise N.S. (2004). Control System Engineering (4th Ed), John Wiley & Sons.
- Chapter 5
  i. Dorf R.C., Bishop R.H. (2001). Modern Control Systems (9th Ed), Prentice Hall.